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Irreducible decomposition for tensor product representations of Jordanian quantum algebras

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Abstract. Tensor products of irreducible representations of the Jordanian quantum algebras $\mathcal{U}_h(sl(2))$ and $\mathcal{U}_h(su(1,1))$ are considered. For both the highest weight finite-dimensional representations of $\mathcal{U}_h(sl(2))$ and the lowest weight infinite-dimensional ones of $\mathcal{U}_h(su(1,1))$, it is shown that tensor product representations are reducible and that the decomposition rules to irreducible representations are exactly the same as those of corresponding Lie algebras.

1. Introduction

Recent works on quantum matrices in two dimensions [1, 2] have introduced a new deformation of the Lie algebra $sl(2)$ called h-deformation or Jordanian deformation $\mathcal{U}_h(sl(2))$ [3]. Some algebraic aspects of $\mathcal{U}_h(sl(2))$ have been investigated and it has been shown that $\mathcal{U}_h(sl(2))$ is a quasitriangular Hopf algebra [4, 5] and that $\mathcal{U}_h(sl(2))$ can be constructed from the Drinfelf–Jimbo deformation by a contraction [6]. Furthermore, two kinds of nonlinear relations between the generators of $sl(2)$ and $\mathcal{U}_h(sl(2))$ have been obtained [7, 8].

On the other hand, representation theories of $\mathcal{U}_h(sl(2))$ have not been well developed yet. What we know so far is that the finite-dimensional irreducible representations of $\mathcal{U}_h(sl(2))$ are classified in exactly the same way as those of $sl(2)$. To show this, the standard singular vector construction method was used in [9, 10]. The authors of [8] used the nonlinear relation between the generators of $sl(2)$ and $\mathcal{U}_h(sl(2))$, while boson realizations of $\mathcal{U}_h(sl(2))$ were used in [11]. In [11], it was shown that decomposition rules of tensor product representations are the same as $sl(2)$ for some low-dimensional representations.

In this paper, we consider the irreducible decomposition for tensor product representations of Jordanian quantum algebras. Representations discussed in this paper are the highest weight finite-dimensional ones for $\mathcal{U}_h(sl(2))$ and the lowest weight infinite-dimensional ones for $\mathcal{U}_h(su(1,1))$. The Jordanian quantum algebra $\mathcal{U}_h(su(1,1))$ is introduced as an algebra being isomorphic to $\mathcal{U}_h(sl(2))$. It is shown that the decomposition rules for both cases are the same as their classical counterparts. Some examples are shown for $\mathcal{U}_h(sl(2))$ in order to discuss explicit expressions of Clebsch–Gordan coefficients. This work is motivated by the fact that well developed representation theories are necessary when we consider physical applications of algebraic objects.

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2. $\mathcal{U}_h(sl(2))$ and its representations

The Jordanian quantum algebra $\mathcal{U}_h(sl(2))$ is an associative algebra with 1 generated by X, Y and H. Their commutation relations are given by [3]

$$\begin{aligned} [H, X] &= 2\frac{\sinh hX}{h} \\ [H, Y] &= -Y(\cosh hX) - (\cosh hX)Y \\ [X, Y] &= H \end{aligned} \tag{2.1}$$

where h is the deformation parameter. The Casimir element is

$$C = \frac{1}{2h}\{Y(\sinh hX) + (\sinh hX)Y\} + \frac{1}{4}H^2 + \frac{1}{4}(\sinh hX)^2. \tag{2.2}$$

In the limit of $h \longrightarrow 0$, $\mathcal{U}_h(sl(2))$ reduces to $sl(2)$. The Hopf algebra structure reads

$$\begin{aligned} &\Delta(X) = X \otimes 1 + 1 \otimes X \\ &\Delta(Y) = Y \otimes \mathrm{e}^{hX} + \mathrm{e}^{-hX} \otimes Y \\ &\Delta(H) = H \otimes \mathrm{e}^{hX} + \mathrm{e}^{-hX} \otimes H \\ &\epsilon(X) = \epsilon(Y) = \epsilon(H) = 0 \\ &S(X) = -X \\ &S(Y) = -\mathrm{e}^{hX} Y \mathrm{e}^{-hX} \\ &S(H) = -\mathrm{e}^{hX} H \mathrm{e}^{-hX}. \end{aligned} \tag{2.3}$$

The finite-dimensional highest weight representations can be easily obtained by using the nonlinear relation between the generators of $sl(2)$ and $\mathcal{U}_h(sl(2))$ given in [8]. Let us define the following elements according to [8]

$$\begin{aligned} Z_+ &= \frac{2}{h}\tanh\frac{hX}{2} \\ Z_- &= \left(\cosh\frac{hX}{2}\right) Y \left(\cosh\frac{hX}{2}\right) \end{aligned} \tag{2.4}$$

then it is not difficult to directly verify that $Z_\pm$ and H satisfy the $sl(2)$ commutation relations

$$[H, Z_\pm] = \pm 2Z_\pm \qquad [Z_+, Z_-] = H \tag{2.5}$$

and the Casimir element yields

$$C = Z_+Z_- + \frac{H}{2}\left(\frac{H}{2} - 1\right) \tag{2.6}$$

by using the identities proved by the mathematical induction

$$\begin{aligned} [H, X^n] &= 2nX^{n-1}\frac{\sinh hX}{h} \\ [Y, X^n] &= -nX^{n-1}H - n(n-1)X^{n-2}\frac{\sinh hX}{h} \end{aligned} \tag{2.7}$$

where n is a natural number. The authors of [8] regarded $Z_\pm, H$ as elements of $sl(2)$, however it is more convenient to regard them as elements of $\mathcal{U}_h(sl(2))$ for our purpose. Namely, their coproducts are given in terms of $\Delta(X)$, $\Delta(Y)$ and $\Delta(H)$.

From (2.5) and (2.6), it is obvious that we can take the following as the irreducible highest weight representations of $\mathcal{U}_h(sl(2))$

$$\begin{aligned} Z_\pm\,|jm\rangle &= \sqrt{(j\mp m)(j\pm m+1)}\,|jm\pm 1\rangle \\ H\,|jm\rangle &= 2m\,|jm\rangle \end{aligned} \tag{2.8}$$

and the eigenvalues of the Casimir element are

$$C\,|jm\rangle = j(j+1)\,|jm\rangle \tag{2.9}$$

where j is a half-integer or an integer and $m=-j,-j+1,\cdots j$, i.e. the usual representation of $sl(2)$. The representation matrices for $X,\ Y$ can be obtained by solving (2.4) with respect to $X,\ Y$ [8].

3. Decomposition rule for $\mathcal{U}_h(sl(2))$

Let us consider the irreducible decomposition of tensor product of two representations specified by the highest weights j_1 and j_2 ; $\{|j_1m_1\rangle\otimes|j_2m_2\rangle\ |m_i=-j_i,-j_i+1,\cdots,j_i,\ i=1,2\}$. Note that a vector $|j_1m_1\rangle\otimes|j_2m_2\rangle$ is no longer an eigenvector of $\Delta(H)$, since $\Delta(H)$ is not given by the classical form $\Delta(H)=H\otimes 1+1\otimes H$. The key of deriving a decomposition rule is to construct the eigenvectors of $\Delta(H)$, since if we obtain such vectors, the decomposition rules can be derived by the same discussion as in the case of $sl(2)$ as we shall see later.

First, we rewrite $\Delta(H)$ in terms of H and $Z_\pm$. From (2.4)

$$\begin{aligned} \mathrm{e}^{hX} &= \frac{1+\frac{hZ_+}{2}}{1-\frac{hZ_+}{2}} = 1+2\sum_{n=1}^{\infty}\left(\frac{hZ_+}{2}\right)^n \\ \mathrm{e}^{-hX} &= \frac{1-\frac{hZ_+}{2}}{1+\frac{hZ_+}{2}} = 1+2\sum_{n=1}^{\infty}\left(-\frac{hZ_+}{2}\right)^n \end{aligned} \tag{3.1}$$

we obtain

$$\Delta(H) = H\otimes 1 + 1\otimes H + H\otimes 2\sum_{n=1}^{\infty}\left(\frac{hZ_+}{2}\right)^n + 2\sum_{n=1}^{\infty}\left(-\frac{hZ_+}{2}\right)^n\otimes H. \tag{3.2}$$

Therefore, for a given vector $|j_1m_1\rangle\otimes|j_2m_2\rangle$, an eigenvector of $\Delta(H)$ may be written as

$$|(j_1m_1)(j_2m_2)\rangle = \sum_{k=0}^{j_1-m_1}\sum_{l=0}^{j_2-m_2}\alpha_{k,l}^{m_1,m_2}\,|j_1m_1+k\rangle\otimes|j_2m_2+l\rangle \tag{3.3}$$

where the coefficients $\alpha_{k,l}^{m_1,m_2}$ are required to be $\alpha_{0,0}^{m_1,m_2}=1$ so as to reproduce the correct limit of $h\longrightarrow 0$. We further require that the eigenvalue of $\Delta(H)$ for the vector (3.3) is $2(m_1+m_2)$. Substituting (3.2) and (3.3) into

$$\Delta(H)\,|(j_1m_1)(j_2m_2)\rangle = 2(m_1+m_2)\,|(j_1m_1)(j_2m_2)\rangle \tag{3.4}$$

then changing summing indices, we obtain

$$\Bigg(\sum_{k=0}^{j_1-m_1}\sum_{l=0}^{j_2-m_2}(k+l)\alpha_{k,l}^{m_1,m_2} + \sum_{k=0}^{j_1-m_1}\sum_{l=1}^{j_2-m_2}\sum_{n=1}^{l}\left(\frac{h}{2}\right)^n 2(m_1+k)$$
$$\times\left\{\frac{(j_2+m_2+l)!(j_2-m_2-l+n)!}{(j_2-m_2-1)!(j_2+m_2+l-n)!}\right\}^{1/2}\alpha_{k,l-n}^{m_1,m_2} + \sum_{k=1}^{j_1-m_1}\sum_{l=0}^{j_2-m_2}\sum_{n=1}^{k}\left(-\frac{h}{2}\right)^n$$

$$\times 2(m_2+l)\left\{\frac{(j_1+m_1+k)!(j_1-m_1-k+n)!}{(j_1-m_1-k)!(j_1+m_1+k-n)!}\right\}^{1/2}\alpha^{m_1,m_2}_{k-n,l}\Bigg)$$
$$\times |j_1 m_1+k\rangle\otimes|j_2 m_2+l\rangle = 0. \tag{3.5}$$

Therefore, $\alpha^{m_1,m_2}_{k,l}$ must satisfy the recurrence relation

$$(k+l)\alpha^{m_1,m_2}_{k,l} + 2(m_1+k)\sum_{n=1}^{l}\left(\frac{h}{2}\right)^n\left\{\frac{(j_2+m_2+l)!(j_2-m_2-l+n)!}{(j_2-m_2-1)!(j_2+m_2+l-n)!}\right\}^{1/2}\alpha^{m_1,m_2}_{k,l-n}$$
$$+2(m_2+l)\sum_{n=1}^{k}\left(-\frac{h}{2}\right)^n\left\{\frac{(j_1+m_1+k)!(j_1-m_1-k+n)!}{(j_1-m_1-k)!(j_1+m_1+k-n)!}\right\}^{1/2}\alpha^{m_1,m_2}_{k-n,l} = 0. \tag{3.6}$$

Next, we rewrite recurrence relation (3.6) in a simpler form. Multiplying (3.6) by $-h/2$ and replacing k with $k-1$, then multiplying it by $\sqrt{(j_1+m_1+k)(j_1-m_1-k+1)}$ and subtracting from (3.6), the obtained relation reads

$$(k+l)\alpha^{m_1,m_2}_{k,l} - \frac{h}{2}\sqrt{(j_1+m_1+k)(j_1-m_1-k+1)}(2m_2+1-k+l)\alpha^{m_1,m_2}_{k-1,l}$$
$$+2\sum_{n=1}^{l}\left(\frac{h}{2}\right)^n\left\{\frac{(j_2+m_2+l)!(j_2-m_2-l+n)!}{(j_2-m_2-l)!(j_2+m_2+l-n)!}\right\}^{1/2}\{(m_1+k)\alpha^{m_1,m_2}_{k,l-n}$$
$$+\frac{h}{2}(m_1+k-1)\sqrt{(j_1+m_1+k)(j_1-m_1-k+1)}\alpha^{m_1,m_2}_{k-1,l-n}\} = 0. \tag{3.7}$$

Multiplying (3.7) by $h/2$ and replacing l with $l-1$, then multiplying it by $\sqrt{(j_2+m_2+l)(j_2-m_2-l+1)}$ and subtracting from (3.7), we obtain the simpler form of recurrence relation

$$(k+l)\alpha^{m_1,m_2}_{k,l} - \frac{h}{2}\sqrt{(j_1+m_1+k)(j_1-m_1-k+1)}(2m_2+1-k+l)\alpha^{m_1,m_2}_{k-1,l}$$
$$+\frac{h}{2}\sqrt{(j_2+m_2+l)(j_2-m_2-l+1)}(2m_1+1+k-l)\alpha^{m_1,m_2}_{k,l-1}$$
$$+\left(\frac{h}{2}\right)^2\sqrt{(j_1+m_1+k)(j_1-m_1-k+1)(j_2+m_2+l)(j_2-m_2-l+1)}$$
$$\times(2m_1+2m_2-2+k+l)\alpha^{m_1,m_2}_{k-1,l-1} = 0. \tag{3.8}$$

The solutions of recurrence relation (3.8) are given by

$$\alpha^{m_1,m_2}_{k,l} = (-1)^l\left(\frac{h}{2}\right)^{k+l}\left\{\frac{(j_1-m_1)!(j_2-m_2)!(j_1+m_1+k)!(j_2+m_2+l)!}{(j_1+m_1)!(j_2+m_2)!(j_1-m_1-k)!(j_2-m_2-l)!}\right\}^{1/2}$$
$$\times\sum_{p=0}\binom{2m_1+k-p}{l-p}\binom{2m_1+k-1}{p}\binom{2m_2}{k-p} \tag{3.9}$$

where the sum on p runs as far as all the binomial coefficients are well defined. For the negative values of m_i, the binomial coefficients are rewritten by the formula

$$\binom{m}{l} = (-1)^l\binom{|m|+l-1}{l}. \tag{3.10}$$

Substituting (3.9) into (3.8), it can be verified that (3.9) gives the solutions of the recurrence relation (3.8). We shall briefly sketch the calculation in the appendix, since it is somewhat complicated.

It has been shown that we can construct a unique vector $|(j_1m_1)(j_2m_2)\rangle$ for two given vectors $|j_1m_1\rangle$, $|j_2m_2\rangle$. The rest steps of deriving a decomposition rule for $\mathcal{U}_h(sl(2))$ is the same as in the case of $sl(2)$. We follow the standard textbook of the quantum mechanics [12].

Acting $\Delta(Z_+)$ and $\Delta(Z_-)$ on $|(j_1m_1)(j_2m_2)\rangle$, we obtain a series of vectors which are eigenvectors of $\Delta(H)$ with eigenvalues

$$-2j, \ldots, 2(m-1), 2m, 2(m+1), \ldots, 2j$$

where $m = m_1 + m_2$ and j denotes the highest weight. Let us set $N(j)$ the number of irreducible representations with highest weight j, and $n(m)$ the number of eigenvectors of $\Delta(H)$ with eigenvalue $2m$. The number of degenerate vectors can be written by the number of irreducible representations

$$n(m) = \sum_{j \geqslant |m|} N(j) \tag{3.11}$$

therefore

$$N(m) = n(m) - n(m+1). \tag{3.12}$$

Since $n(m)$ equals the number of pairs (m_1, m_2) satisfying $m = m_1+m_2$, it can be expressed as

$$n(m) = \begin{cases} 0 & \text{for } |m| > j_1 + j_2 \\ j_1 + j_2 + 1 - |m| & \text{for } j_1 + j_2 \geqslant |m| \geqslant |j_1 - j_2| \\ 2j_2 + 1 & \text{for } |j_1 - j_2| \geqslant |m| \geqslant 0. \end{cases} \tag{3.13}$$

Substituting (3.13) into (3.12), we obtain

$$N(m) = \begin{cases} 1 & \text{for } j_1 + j_2 \geqslant |m| \geqslant |j_1 - j_2| \\ 0 & \text{otherwise.} \end{cases} \tag{3.14}$$

Therefore we have proved the fact that a tensor product of two highest weight representations (highest weights are j_1 and j_2) of $\mathcal{U}_h(sl(2))$ is reducible and the irreducible decomposition rule is shown schematically

$$j_1 \otimes j_2 = (j_1 + j_2) \oplus (j_1 + j_2 - 1) \oplus \cdots \oplus |j_1 - j_2|.$$

Furthermore, each irreducible representation contained in a tensor product is multiplicity free.

4. Some examples for $\mathcal{U}_h(sl(2))$

In this section, some explicit examples of irreducible decomposition, namely some Clebsch–Gordan coefficients, are given. To this end, the explicit form of $\Delta(Z_-)$ is needed. Note that the explicit form of $\Delta(Z_+)$ is not necessary, since the vector which is annihilated by $\Delta(X)$ is also annihilated by $\Delta(Z_+)$.

From (2.4),

$$\Delta(Z_-) = \Delta\left(\cosh\frac{hX}{2}\right)\Delta(Y)\Delta\left(\cosh\frac{hX}{2}\right). \tag{4.1}$$

Using

$$\Delta\left(\cosh\frac{hX}{2}\right) = \cosh\frac{hX}{2} \otimes \cosh\frac{hX}{2} + \sinh\frac{hX}{2} \otimes \sinh\frac{hX}{2} \tag{4.2}$$

and (2.9), (3.1), $\Delta(Z_-)$ can be rewritten as

$$\Delta(Z_-) = Z_- \otimes \sum_{n=0}^{\infty}(n+1)\left(\frac{hZ_+}{2}\right)^n + \sum_{n=0}^{\infty}(n+1)\left(-\frac{hZ_+}{2}\right)^n \otimes Z_- + h\left(C - \frac{H^2}{4}\right) \otimes \sum_{m=1}^{\infty} m\left(\frac{hZ_+}{2}\right)^m - \sum_{m=1}^{\infty} m\left(-\frac{hZ_+}{2}\right)^m \otimes h\left(C - \frac{H^2}{4}\right) + \left(\frac{h}{2}\right)^2 Z_+Z_-Z_+ \otimes \sum_{k=2}^{\infty}(k-1)\left(\frac{hZ_+}{2}\right)^k + \sum_{k=2}^{\infty}(k-1)\left(-\frac{hZ_+}{2}\right)^k \otimes \left(\frac{h}{2}\right)^2 Z_+Z_-Z_+. \tag{4.3}$$

We consider the cases of $m = j_1 + j_2,\ j_1 + j_2 - 1$ and $j_1 + j_2 - 2$. Using the result of section 3, the eigenvectors of $\Delta(H)$ with eigenvalues $2m$ are constructed.

(1) $m = j_1 + j_2$

$$|(j_1 j_1)(j_2 j_2)\rangle = |j_1 j_1\rangle \otimes |j_2 j_2\rangle\,. \tag{4.4}$$

(2) $m = j_1 + j_2 - 1$

$$|(j_1 j_1)(j_2 j_2 - 1)\rangle = |j_1 j_1\rangle \otimes |j_2 j_2 - 1\rangle - h j_1 \sqrt{2j_2}\,|j_1 j_1\rangle \otimes |j_2 j_2\rangle \tag{4.5}$$

$$|(j_1 j_1 - 1)(j_2 j_2)\rangle = |j_1 j_1 - 1\rangle \otimes |j_2 j_2\rangle + h j_2 \sqrt{2j_1}\,|j_1 j_1\rangle \otimes |j_2 j_2\rangle\,. \tag{4.6}$$

(3) $m = j_1 + j_2 - 2$

$$|(j_1 j_1)(j_2 j_2 - 2)\rangle = |j_1 j_1\rangle \otimes |j_2 j_2 - 2\rangle - h j_1 \sqrt{2(2j_2 - 1)}\,|j_1 j_1\rangle \otimes |j_2 j_2 - 1\rangle + \frac{h^2}{2} j_1(2j_1 - 1)\sqrt{j_2(2j_2 - 1)}\,|j_1 j_1\rangle \otimes |j_2 j_2\rangle \tag{4.7}$$

$$|(j_1 j_1 - 1)(j_2 j_2 - 1)\rangle = |j_1 j_1 - 1\rangle \otimes |j_2 j_2 - 1\rangle - h(j_1 - 1)\sqrt{2j_2}\,|j_1 j_1 - 1\rangle \otimes |j_2 j_2\rangle + h(j_2 - 1)\sqrt{2j_1}\,|j_1 j_1\rangle \otimes |j_2 j_2 - 1\rangle - h^2(2j_1 j_2 - j_1 - j_2)\sqrt{j_1 j_2}\,|j_1 j_1\rangle \otimes |j_2 j_2\rangle \tag{4.8}$$

$$|(j_1 j_1 - 2)(j_2 j_2)\rangle = |j_1 j_1 - 2\rangle \otimes |j_2 j_2\rangle + h j_2 \sqrt{2(2j_1 - 1)}\,|j_1 j_1 - 1\rangle \otimes |j_2 j_2\rangle + \frac{h^2}{2} j_2(2j_2 - 1)\sqrt{j_1(2j_1 - 1)}\,|j_1 j_1\rangle \otimes |j_2 j_2\rangle\,. \tag{4.9}$$

Let us construct the representation basis with highest weight $j_1 + j_2,\ j_1 + j_2 - 1$ and $j_1 + j_2 - 2$. It is easy to verify

$$\Delta(X)\,|(j_1 j_1)(j_2 j_2)\rangle = 0$$

and

$$\Delta(X)\,|(j_1 j_1 - 1)(j_2 j_2)\rangle = \sqrt{2j_1}\,|(j_1 j_1)(j_2 j_2)\rangle$$
$$\Delta(X)\,|(j_1 j_1)(j_2 j_2 - 1)\rangle = \sqrt{2j_2}\,|(j_1 j_1)(j_2 j_2)\rangle$$

therefore we obtain

$$|j_1 + j_2 \quad j_1 + j_2\rangle = |j_1 j_1\rangle \otimes |j_2 j_2\rangle \tag{4.10}$$

$$|j_1 + j_2 - 1 \quad j_1 + j_2 - 1\rangle = -\sqrt{j_2}\,|(j_1 j_1 - 1)(j_2 j_2)\rangle + \sqrt{j_1}\,|(j_1 j_1)(j_2 j_2 - 1)\rangle\,. \tag{4.11}$$

A similar calculation gives

$$|j_1 + j_2 - 2 \quad j_1 + j_2 - 2\rangle = \sqrt{j_1(2j_1 - 1)}\,|(j_1 j_1)(j_2 j_2 - 2)\rangle - \sqrt{(2j_1 - 1)(2j_2 - 1)}\,|(j_1 j_1 - 1)(j_2 j_2 - 1)\rangle + \sqrt{j_2(2j_2 - 1)}\,|(j_1 j_1 - 2)(j_2 j_2)\rangle\,. \tag{4.12}$$

Other basis vectors are obtained by acting $\Delta(Z_-)$ on the highest weight vectors. They read

$$|j_1+j_2 \quad j_1+j_2-1\rangle = \frac{1}{\sqrt{j_1+j_2}}\left(\sqrt{j_1}\,|(j_1 j_1-1)(j_2 j_2)\rangle + \sqrt{j_2}\,|(j_1 j_1)(j_2 j_2-1)\rangle\right)$$

$$|j_1+j_2 \quad j_1+j_2-2\rangle = \frac{1}{\sqrt{(j_1+j_2)(2j_1+2j_2-1)}}\left\{\sqrt{j_2(2j_2-1)}\,|(j_1 j_1)(j_2 j_2-2)\rangle \right.$$
$$\left. +2\sqrt{j_1 j_2}\,|(j_1 j_1-1)(j_2 j_2-1)\rangle + \sqrt{j_1(2j_1-1)}\,|(j_1 j_1-2)(j_2 j_2)\rangle\right\}$$

$$|j_1+j_2-1 \quad j_1+j_2-2\rangle = \frac{1}{\sqrt{j_1+j_2-1}}\left\{\sqrt{j_1(2j_2-1)}\,|(j_1 j_1)(j_2 j_2-2)\rangle \right.$$
$$\left. +(j_1-j_2)\,|(j_1 j_1-1)(j_2 j_2-1)\rangle - \sqrt{j_2(2j_1-1)}\,|(j_1 j_1-2)(j_2 j_2)\rangle\right\}.$$

It is remarkable that the Clebsch–Gordan coefficients for the vectors $|(j_1 m_1)(j_2 m_2)\rangle$ considered in this section are the same as the classical ones except for the normalization factors, while the Clebsch–Gordan coefficients for the usual tensor products $|j_1 m_1\rangle \otimes |j_2 m_2\rangle$ are deformed. It may be a future work to investigate whether it holds for any values of $j = j_1+j_2, j_1+j_2-1, \ldots, |j_1-j_2|$ and allowed m for each j.

5. $\mathcal{U}_h(su(1,1))$ and its representations

We define $\mathcal{U}_h(su(1,1))$ as an algebra isomorphic to $\mathcal{U}_h(sl(2))$. Denoting the generators of $\mathcal{U}_h(su(1,1))$ by R, V, F, they are defined

$$R = -X \qquad V = Y \qquad F = H. \tag{5.1}$$

This definition is inspired from the isomorphism between $sl(2)$ and $su(1,1)$

$$K_\pm = \mp J_\pm \qquad K_0 = J_0 \tag{5.2}$$

where $J_\pm, J_0$ and $K_\pm, K_0$ are generators of $sl(2)$ and $su(1,1)$ respectively. Combining the isomorphism (5.2) and the nonlinear relation between generators of $sl(2)$ and $\mathcal{U}_h(sl(2))$ [8], the isomorphism (5.1) is obtained.

All algebraic structures of $\mathcal{U}_h(su(1,1))$ can easily be derived by using (5.1). The commutation relations are obtained from (2.1)

$$\begin{aligned}
&[F, R] = 2\frac{\sinh hR}{h} \\
&[F, V] = -V(\cosh hR) - (\cosh hR)V \\
&[R, V] = -F
\end{aligned} \tag{5.3}$$

the Casimir element is from (2.2)

$$C' = -\frac{1}{2h}\{V(\sinh hR) + (\sinh hR)V\} + \frac{1}{4}F^2 + \frac{1}{4}(\sinh hR)^2. \tag{5.4}$$

The Hopf algebra mappings for $\mathcal{U}_h(su(1,1))$ are obtained from (2.3).

Let us next consider representations of $\mathcal{U}_h(su(1,1))$. The strategy is the same as the one for $\mathcal{U}_h(sl(2))$. We define new elements of $\mathcal{U}_h(su(1,1))$

$$T_+ = \frac{2}{h}\tanh\frac{hR}{2} \qquad T_- = \left(\cosh\frac{hR}{2}\right) V \left(\cosh\frac{hR}{2}\right) \tag{5.5}$$

then $T_\pm$ and F satisfy the $su(1,1)$ commutation relations

$$[F, T_\pm] = \pm 2T_\pm \qquad [T_+, T_-] = -F \tag{5.6}$$

and the Casimir element reads

$$C' = \frac{F}{2}\left(\frac{F}{2} - 1\right) - T_+ T_- . \tag{5.7}$$

These are easily verified with the identities

$$\begin{aligned} [F, R^n] &= 2n R^{n-1} \frac{\sinh hR}{h} \\ [V, R^n] &= n R^{n-1} F + n(n-1) R^{n-2} \frac{\sinh hR}{h} . \end{aligned} \tag{5.8}$$

It is now clear that we can take a representation of $su(1,1)$ as the one for $T_\pm, F$. In this paper, we concentrate on the representation called the positive discrete series [13, 14] which is a lowest weight infinite-dimensional representation

$$\begin{aligned} F\,|\kappa\mu\rangle &= 2\mu\,|\kappa\mu\rangle \\ T_\pm\,|\kappa\mu\rangle &= \sqrt{(\mu \pm \kappa)(\mu \mp \kappa \pm 1)}\,|\kappa\mu \pm 1\rangle \end{aligned} \tag{5.9}$$

and the eigenvalue of the Casimir element is given by

$$C'\,|\kappa\mu\rangle = \kappa(\kappa - 1)\,|\kappa\mu\rangle \tag{5.10}$$

where κ can take any positive value and $\mu = \kappa, \kappa+1, \kappa+2, \ldots$. The representation matrices for R, V can be obtained by solving (5.5) with respect to R, V.

6. Decomposition rule for $\mathcal{U}_h(su(1,1))$

In this section, we show that a decomposition rule of the product of two positive discrete series of $\mathcal{U}_h(su(1,1))$ is the same as $su(1,1)$. We consider a tensor product representation of positive discrete series with the lowest weight κ_1, κ_2. Using (5.5), the coproduct of F can be rewritten as

$$\Delta(F) = F \otimes 1 + 1 \otimes F + F \otimes 2\sum_{n=1}^{\infty}\left(-\frac{hT_+}{2}\right)^n + 2\sum_{n=1}^{\infty}\left(\frac{hT_+}{2}\right)^n \otimes F. \tag{6.1}$$

For a given vector $|(\kappa_1\mu_1)(\kappa_2\mu_2)\rangle$, the eigenvector of $\Delta(F)$ may be written

$$|(\kappa_1\mu_1)(\kappa_2\mu_2)\rangle\rangle = \sum_{\rho,\sigma=0}^{\infty} \alpha^{\mu_1,\mu_2}_{\rho,\sigma}\,|\kappa_1\mu_1 + \rho\rangle \otimes |\kappa_2\mu_2 + \sigma\rangle . \tag{6.2}$$

We require that the eigenvalue of $\Delta(F)$ for the vector (6.2) is $2(\mu_1 + \mu_2)$. Because of the consistency with the limit of $h \longrightarrow 0$, we set $\alpha^{\mu_1,\mu_2}_{0,0} = 1$. Substituting (6.1) and (6.2) into the relation $\Delta(F)\,|(\kappa_1\mu_1)(\kappa_2\mu_2)\rangle\rangle = 2(\mu_1+\mu_2)\,|(\kappa_1\mu_1)(\kappa_2\mu_2)\rangle\rangle$, we obtain the recurrence relation for $\alpha^{\mu_1,\mu_2}_{\rho,\sigma}$

$$\begin{aligned} &(\rho+\sigma)\alpha^{\mu_1,\mu_2}_{\rho,\sigma} + 2(\mu_1+\rho)\sum_{n=1}^{\sigma}\left(-\frac{h}{2}\right)^n \left\{\frac{(\mu_2+\sigma+\kappa_2-1)!(\mu_2+\sigma-\kappa_2)!}{(\mu_2+\sigma+\kappa_2-1-n)!(\mu_2+\sigma-\kappa_2-n)!}\right\}^{1/2} \\ &\qquad \times \alpha^{\mu_1,\mu_2}_{\rho,\sigma-n} + 2(\mu_2+\sigma)\sum_{n=1}^{\rho}\left(\frac{h}{2}\right)^n \\ &\qquad \times \left\{\frac{(\mu_1+\rho+\kappa_1-1)!(\mu_1+\rho-\kappa_1)!}{(\mu_1+\rho+\kappa_1-1-n)!(\mu_1+\rho+\kappa_1-n)!}\right\}^{1/2} \alpha^{\mu_1,\mu_2}_{\rho-n,\sigma} = 0. \end{aligned} \tag{6.3}$$

Repeating the same procedure as the case of $\mathcal{U}_h(sl(2))$, the recurrence relation (6.3) is rewritten in the simpler form

$$\begin{aligned}(\rho+\sigma)\alpha^{\mu_1,\mu_2}_{\rho,\sigma} &- \frac{h}{2}\sqrt{(\mu_2+\sigma+\kappa_2-1)(\mu_2+\sigma-\kappa_2)}(2\mu_1+1+\rho-\sigma)\alpha^{\mu_1,\mu_2}_{\rho,\sigma-1}\\ &+\frac{h}{2}\sqrt{(\mu_1+\rho+\kappa_1-1)(\mu_1+\rho-\kappa_1)}(2\mu_2+1-\rho+\sigma)\alpha^{\mu_1,\mu_2}_{\rho-1,\sigma}\\ &+\left(\frac{h}{2}\right)^2\sqrt{(\mu_1+\rho+\kappa_1-1)(\mu_1+\rho-\kappa_1)(\mu_2+\sigma+\kappa_2-1)(\mu_2+\sigma-\kappa_2)}\\ &\times(2\mu_1+2\mu_2-2+\rho+\sigma)\alpha^{\mu_1,\mu_2}_{\rho-1,\sigma-1}=0. \end{aligned} \tag{6.4}$$

The solutions of (6.4) are given by

$$\begin{aligned}\alpha^{\mu_1,\mu_2}_{\rho,\sigma} &= (-1)^{\rho}\left(\frac{h}{2}\right)^{\rho+\sigma}\\ &\times\left\{\frac{(\mu_1-\kappa_1+\rho)!(\mu_1+\kappa_1-1+\rho)!(\mu_2-\kappa_2+\sigma)!(\mu_2+\kappa_2-1+\sigma)!}{(\mu_1-\kappa_1)!(\mu_1+\kappa_1-1)!(\mu_2-\kappa_2)!(\mu_2+\kappa_2-1)!}\right\}^{1/2}\\ &\times\sum_{p=0}\binom{2\mu_1+\rho-p}{\sigma-p}\binom{2\mu_1+\rho-1}{p}\binom{2\mu_2}{\rho-p}\end{aligned} \tag{6.5}$$

where the sum on p runs as far as all the binomial coefficients are well defined. It can be proved that (6.5) satisfies the recurrence relation (6.4) in the same way as in the appendix.

It has been shown that we can construct a unique eigenvector of $\Delta(F)$ with eigenvalue $2(\mu_1+\mu_2)$ for given vectors $|\kappa_1\mu_1\rangle$, $|\kappa_2\mu_2\rangle$. Acting $\Delta(T_\pm)$ on $|(\kappa_1\mu_1)(\kappa_2\mu_2)\rangle$, we can construct a series of eigenvectors of $\Delta(F)$ with eigenvalues

$$\kappa,\kappa+1,\ldots,\mu,\mu+1,\ldots$$

where $\mu=\mu_1+\mu_2$ and it is clear that the lowest possible value of μ (denoted by κ) is $\kappa_1+\kappa_2$. Let us set $N(\kappa)$ the number of irreducible representations with lowest weight κ, and $n(\mu)$ the number of eigenvectors of $\Delta(F)$ with eigenvalue 2μ. The number of degenerate vectors can be written by the number of irreducible representation

$$n(\mu)=\sum_{\kappa\leqslant\mu}N(\kappa) \tag{6.6}$$

therefore

$$N(\mu)=n(\mu)-n(\mu-1). \tag{6.7}$$

Since $n(\mu)$ equals the number of pairs (μ_1,μ_2) satisfying $\mu=\mu_1+\mu_2$, it is given by

$$n(\mu)=\begin{cases}0 & \text{for } \mu<\kappa_1+\kappa_2\\ \mu-\kappa_1-\kappa_2+1 & \text{for } \mu\geqslant\kappa_1+\kappa_2.\end{cases} \tag{6.8}$$

Substituting (6.8) into (6.7),

$$N(\mu)=\begin{cases}0 & \text{for } \mu<\kappa_1+\kappa_2\\ 1 & \text{for } \mu\geqslant\kappa_1+\kappa_2.\end{cases} \tag{6.9}$$

Therefore we have proved the fact that a tensor product of two positive discrete series of $\mathcal{U}_h(su(1,1))$ is reducible and the irreducible decomposition rule is given schematically by

$$\kappa_1\otimes\kappa_2=\kappa_1+\kappa_2\oplus\kappa_1+\kappa_2+1\oplus\ \kappa_1+\kappa_2+2\oplus\cdots.$$

Furthermore, each irreducible representation contained in the tensor product is multiplicity free.

7. Conclusion

We have shown that, for both highest weight finite-dimensional representations of $\mathcal{U}_h(sl(2))$ and lowest weight infinite-dimensional ones of $\mathcal{U}_h(su(1,1))$, tensor product representations are reducible and the decomposition rules to irreducible representations are exactly the same as those of the corresponding Lie algebras. We concentrate on the positive discrete series of $\mathcal{U}_h(su(1,1))$, the same result may hold for the negative discrete series which are highest weight infinite-dimensional representations, since the difference between positive and negative discrete series is to use highest weight or lowest one. The Lie algebra $su(1,1)$ has two other infinite-dimensional representations [13]. The corresponding representations of $\mathcal{U}_h(su(1,1))$ may obtain the inverse mapping of (5.5), however, tensor products of such representations are still an open problem.

The construction of eigenvectors of $\Delta(H)$ and $\Delta(F)$ is the key of the proof. The other steps of the proof are nothing but those for the Lie algebras. This parallelism in the representation theories between Jordanian quantum algebras and the corresponding Lie algebras may suggest further similarities. For example, we might be able to obtain the Clebsch–Gordan coefficients by the same method as the classical case, Racha–Wigner type of calculus ($6\,j$, $9\,j$ symbols, tensor operators, Wigner–Eckart's theorem etc) might be possible for the Jordanian quantum algebras. The similarity in the representation theories may also suggest that the Jordanian quantum algebras are applicable to various fields in physics. These will be future works.

Appendix

In this appendix, we show that (3.9) is the solution of recurrence relation (3.8). Substituting (3.9) into (3.8), then using the identities

$$(2m_1+1+k-l)\binom{2m_1+k-p}{l-1-p} = (l-p)\binom{2m_1+k-p}{l-p}$$
$$(2m_2+1-k+p)\binom{2m_2}{k-1-p} = (k-p)\binom{2m_2}{k-p}$$

the left-hand side of the recurrence relation (3.8), up to a factor of

$$(-1)^l\left(\frac{h}{2}\right)^{k+l}\left\{\frac{(j_1-m_1)!(j_2-m_2)!(j_1+m_1+k)!(j_2+m_2+l)!}{(j_1+m_1)!(j_2+m_2)!(j_1-m_1-k)!(j_2-m_2-l)!}\right\}^{1/2}$$

can be rewritten

$$\begin{aligned}
k\sum_{p=0}&\binom{2m_1+k-p}{l-p}\binom{2m_1+k-1}{p}\binom{2m_2}{k-p}\\
&-\sum_{p=0}(k-p)\binom{2m_1+k-1-p}{l-p}\binom{2m_1+k-2}{p}\binom{2m_2}{k-p}\\
&-\sum_{p=0}(l-p)\binom{2m_1+k-1-p}{l-p}\binom{2m_1+k-2}{p}\binom{2m_2}{k-1-p}\\
&+\sum_{p=0}p\binom{2m_1+k-p}{l-p}\binom{2m_1+k-1}{p}\binom{2m_2}{k-p}\\
&-\sum_{p=0}(k-p)\binom{2m_1+k-1-p}{l-1-p}\binom{2m_1+k-2}{p}\binom{2m_2}{k-p}
\end{aligned}$$

$$
-\sum_{p=0}(2m_1-3+2k+l-p)\binom{2m_1+k-1-p}{l-1-p}
\times\binom{2m_1+k-2}{p}\binom{2m_2}{k-1-p}.
$$

Redefining $p+1$ as p in the third and the sixth summation, the fourth and the sixth summation can be combined. The second and the fifth summation can also be combined by using the identity

$$
\binom{n}{l-1}+\binom{n}{l}=\binom{n+1}{l}.
$$

At this stage, the left-hand side of (3.8) reads

$$
\begin{aligned}
k\sum_{p=0}&\binom{2m_1+k-p}{l-p}\binom{2m_1+k-1}{p}\binom{2m_2}{k-p}\\
&-\sum_{p=0}(k-p)\binom{2m_1+k-p}{l-p}\binom{2m_1+k-2}{p}\binom{2m_2}{k-p}\\
&-\sum_{p=1}(l-p+1)\binom{2m_1+k-p}{l+1-p}\binom{2m_1+k-2}{p-1}\binom{2m_2}{k-p}\\
&-\sum_{p=1}(k+l-1-p)\binom{2m_1+k-p}{l-p}\binom{2m_1+k-2}{p-1}\binom{2m_2}{k-p}.
\end{aligned}
$$

It is now easy to see that this always vanishes, noting that the last two summations are combined to give

$$
\sum_{p=1}(2m_1+2k-1-p)\binom{2m_1+k-p}{l-p}\binom{2m_1+k-2}{p-1}\binom{2m_2}{k-p}.
$$

This completes the proof.

Note added in proof. After this manuscript was submitted we received a preprint [15], where the explicit formulae for the $\mathcal{U}_h(sl(2))$ Clebsch–Gordan coefficients were obtained and a solution to the question presented at the end of section 4 was given.

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